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A SUMMATION FORMULAE FOR BILATERAL BASIC HYPERGEOMETRIC SERIES

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ABSTRACT

In this paper, making use of decomposition of series method, and knowing summation formulae for truncated series to drive certain interesting transformation involving bilateral basic hypergeometric series.

Keywords: Basic hypergeometric series, truncated series, decomposition method and summation formula.

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1. Introduction, Notation and Definitions.

Since the times of Ramannjan's and Bailey's have been playing a very important role in the field of Bilateral basic hypergeometric series. This part of Ramannjan's and Bailey's work have been treated and developed consequently by several authors Aqarual, R. P. [1], Andrew, G. E. [2] , Singh, S. [5], etc.

Recently, Vasuki, K. R. and Chammaraju, C. [6], Zhang, Z. and Qiuxia, H. [8], Pankaj, S., Tunis, E., Ajit, P. and Mahmoud, E. [4] have derived an alternative proof of Bilateral basic hypergeometric series by series transformation.

We have some notation and definition from Gasper G. and Rahman M. [3], for a , q are real or complex and $|q| < 1$, $q-$shifted factorial is defined by

$$(a;q)_n = \begin{cases} (1-a)(1-aq)\dots(1-aq^{n-1}), & ;n>0 \\ 1 & ;n=0. \end{cases} \tag{1.1}$$

and

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k), \tag{1.2}$$

We define a basic hypergeometric series,

$${}_r\phi_s\left[\, a_1,\dots,a_r; b_1,\dots,b_s \;;q,z\right] = {}_r\phi_s\left[\begin{matrix} a_1,a_2,\dots,a_r \\ b_1,b_2,\dots,b_s \end{matrix}\;;q,z\right]$$

$$= \sum_{n=0}^{\infty}\frac{(a_1;q)_n(a_2;q)_n\dots(a_r;q)_n\; q^{\lambda n(n+1)}}{(q;q)_n(b_1;q)_n(b_2;q)_n\dots(b_s;q)_n}z^n, \quad (|z|<1). \tag{1.3}$$

We, also, define a truncated series,

$${}_r\phi_s\left[\begin{matrix} a_1,a_2,\dots,a_r \\ b_1,b_2,\dots,b_s \end{matrix}\;;q,z\right]_N = \sum_{n=0}^{\infty}\frac{(a_1;q)_n(a_2;q)_n\dots(a_r;q)_n\; q^{\lambda n(n+1)/2}}{(q;q)_n(b_1;q)_n(b_2;q)_n\dots(b_s;q)_n}z^n,. \tag{1.4}$$

and bilateral basic hypergeometric series is defined by

$${}_r\psi_r\left[\, a_1,\dots,a_r; b_1,\dots,b_s \;;q,z\right] = \sum_{n=-\infty}^{\infty}\frac{(a_1;q)_n(a_2;q)_n\dots(a_r;q)_n\; q^{\lambda n(n+1)/2}}{(q;q)_n(b_1;q)_n(b_2;q)_n\dots(b_r;q)_n}z^n, \quad (|z|<1). \tag{1.5}$$

During the process in this paper, we also make use of the following notations as

$$[a;q]_{-n}=\frac{(-q/a;q)_n\ q^{\binom{n}{2}-nk}}{(q/a;q)_n} \tag{1.6}$$

and

$$[a;q]_{2n}=(a;q^2)_n(aq;q^2)_n \tag{1.7}$$

We shall need the following known results in our analysis,

$$ {}_4\phi_3\left[\begin{matrix} a,c,aq^{3/2+n}/c,q^{-n} \\ aq/c,cq^{-n-(1/2)},aq^{2+n}\end{matrix};q,q\right]=\frac{[q/c\sqrt{a};q]_{n+1}[\sqrt{aq}/c;q]_{n+1}[\sqrt{q};q^{1/2}]_{2n+1}[a;q]_{n+2}}{2\sqrt{a}[aq/c;q]_{n+1}[\sqrt{q}/c;q]_{n+1}[q;q]_{n+1}[\sqrt{a};q^{1/2}]_{2n+2}}-$$

$$\frac{[-q/c\sqrt{a};q]_{n+1}[\sqrt{-aq}/c;q]_{n+1}[\sqrt{q};q^{1/2}]_{2n+1}[a;q]_{n+2}}{-2\sqrt{a}[ca/b;q]_{n+1}[\sqrt{q}/c;q]_{n+1}[q;q]_{n+1}[-\sqrt{a};q^{1/2}]_{2n+2}} \tag{1.8}$$

[7], Eq. (4.8), p.78

$$ {}_4\phi_3\left[\begin{matrix} a,-qa^{1/2},b,q^{-n} \\ -a^{1/2},aq/b,aq^{1+n}\end{matrix};q,\frac{q^{1+n}a^{1/2}}{b}\right]=\frac{[aq,qa^{1/2}/b;q]_n}{[qa^{1/2},aq/b;q]_n} \tag{1.9}$$

[3], App..(II.14),pp. 237

$$ {}_3\phi_2\left[\begin{matrix} q^{-n},a,b \\ d,e\end{matrix};q,q\right]=\frac{[e/a;q]_n a^n}{[e;q]_n}\ {}_3\phi_2\left[\begin{matrix} q^{-n},a,d/b \\ d,q^{1-n}a/e\end{matrix};q,bq/e\right] \tag{1.10}$$

[2], Eq. (10.10.5), p.525]

$$ {}_3\phi_2\left[\begin{matrix} a,q\sqrt{a},q^{-n} \\ \sqrt{a},aq^{n+1}\end{matrix};q,-q^n\right]=\frac{[aq;q]_n[-1;q]_n(1+\sqrt{a})}{2[aq;q^2]_n}+\frac{[aq;q]_n[-1;q]_n(1+\sqrt{a})}{2[\sqrt{a};q]_n[-q\sqrt{a};q]_n} \tag{1.11}$$

[7], Eq. (4.2), p.76

$$ {}_2\phi_1\left[\begin{matrix} a,q^{-n} \\ aq^n\end{matrix};q,-q^{n+1/2}\right]=\frac{[a;q]_n[-\sqrt{q};q]_n}{2[\sqrt{-aq};q]_n[\sqrt{a};q]_n}+\frac{[a;q]_n[-\sqrt{q};q]_n}{2[\sqrt{-a};q]_n[\sqrt{aq};q]_n} \tag{1.12}$$

[7], Eq. (4.17), p.80])

2. Main Results .

$$
{}_6\psi_6\left[\begin{matrix} c, cq, q^{5/2+n}/c, q^{7/2+n}/c, q^{-n}, q^{-n+1} \\ q^2/c, q^3/c, cq^{-n-(1/2)}, cq^{-n+(1/2)}, q^{3+n}, q^{4+n} \end{matrix} ; q^2, q^2\right] =
$$

$$
\frac{[q^{3/2}/c;q]_{n+1}[q/c;q]_{n+1}[\sqrt{q};q^{1/2}]_{2n+1}[q;q]_{n+2}}{2\sqrt{q}[q^2/c;q]_{n+1}[\sqrt{q}/c;q]_{n+1}[q;q]_{n+1}[\sqrt{q};q^{1/2}]_{2n+2}} -
$$

$$
\frac{[-q^3/2/c;q]_{n+1}[-q/c;q]_{n+1}[\sqrt{q};q^{1/2}]_{2n+1}[q;q]_{n+2}}{-2\sqrt{q}[cq/b;q]_{n+1}[\sqrt{q}/c;q]_{n+1}[q;q]_{n+1}[-\sqrt{q};q^{1/2}]_{2n+2}} \tag{2.1}
$$

$$
{}_5\psi_5\left[\begin{matrix} -q^{5/2}, b, bq, q^{-n}, q^{-n+1} \\ -q^{1/2}, q^2/b, q^3/b, q^{2+n}, q^{3+n} \end{matrix} ; q^2, \frac{q^{3+2n}}{b}\right] = \frac{[q^2, q^{3/2}/b;q]_n}{[q^{3/2}, q^2/b;q]_n} \tag{2.2}
$$

$$
{}_4\psi_4\left[\begin{matrix} b, bq, q^{-n}, q^{-n+1} \\ d, dq, e, eq \end{matrix} ; q^2, q^2\right] = \frac{[e/q;q]_n q^n}{[e;q]_n}\, {}_2\psi_2\left[\begin{matrix} q^{-n}, d/b \\ d, q^{2-n}/e \end{matrix} ; q, bq/e\right] \tag{2.3}
$$

$$
{}_3\psi_3\left[\begin{matrix} q^{5/2}, q^{-n}, q^{-n+1} \\ \sqrt{q}, q^{n+2}, q^{n+3} \end{matrix} ; q^2, -q^{2n}\right] = \frac{[q^2;q]_n[-1;q]_n(1+\sqrt{q})}{2[-q;q]_n} + \frac{[q^2;q]_n[-1;q]_n(1+\sqrt{q})}{2[\sqrt{q};q]_n[-q^{3/2};q]_n} \tag{2.4}
$$

3. proof of Main Results .

As an illustration to establish (2.1), we first write out the left-hand side of (1.8) as follows:

$$
\sum_{r=0}^{n} \frac{(a, c, a/c\; q^{3/2+n}, q^{-n})_r}{(aq/c, cq^{-n-1/2}, aq^{2+n};q)_r}(q)^r \tag{3.1}
$$

Now, letting $a = q$ in (3.1) and after simplification, we get

$$= \sum_{r=0}^{[n/2]} \frac{(c, q^{5/2+n}/c\ , q^{-n})_{2r}}{(q^2/c, cq^{-n-1/2}, q^{3+n}; q)_{2r}} (q)^{2r} + \sum_{r=1}^{[n/2]} \frac{(c, q^{5/2+n}/c\ , q^{-n})_{2r-1}}{(q^2/c, cq^{-n-1/2}, q^{3+n}; q)_{2r-1}} (q)^{2r-1} \tag{3.2}$$

$$= \sum_{r=0}^{[n/2]} \frac{(c, q^{5/2+n}/c\ , q^{-n})_{2r}}{(q^2/c, cq^{-n-1/2}, q^{3+n}; q)_{2r}} (q^2)^{r} + \sum_{r=[-n/2]}^{-1} \frac{(c, q^{5/2+n}/c, q^{-n})_{2r}}{(q^2/c, cq^{-n-1/2}, q^{3+n}; q)_{2r}} (q^2)^{r} \tag{3.3}$$

$$= \sum_{r=-[n/2]}^{[n/2]} \frac{(c, q^{5/2+n}/c\ , q^{-n})_{2r}}{(q^2/c, cq^{-n-1/2}, q^{3+n}; q)_{2r}} (q^2)^{r} \tag{3.4}$$

with the help of (1.7) in (3.4) after simplification, we get the required result (2.1) In similar way we have also proved (2.2, 2.3, 2.4)

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